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The tight orthogonal homotopic bases of closed oriented triangulated surfaces and their computing

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ABSTRACT

In this paper, for a closed oriented triangulated surface with genus g, a method with $\mathcal{O}(g^3 n \log n)$ running time of constructing tight orthogonal homotopic bases is presented, where a tight orthogonal homotopic basis is a homotopic basis with the properties: 1. the elements of this basis are cycles, 2. any two adjacent cycles of this basis have exactly one common point, 3. any two nonadjacent cycles of this basis have no common point, and 4. any cycle of this basis is one of the shortest cycles of its homotopic group. The major difference between orthogonal homotopic bases and the well-known canonical homotopic bases is that all the cycles of a canonical homotopic basis have a common point and there is no other common point between any two cycles of the canonical homotopic basis while any two adjacent cycles of an orthogonal homotopic basis have exactly one common point and there is no common point among any three cycles of this basis.

1. Introduction

A triangulated surface, triangulated manifold or triangular mesh is a 2-manifold that is composed of triangles such that the intersection between any two triangles is their common vertex or their common edge if the intersection is not empty. A triangulated surface can have boundary. But, in this paper we mainly discuss bounded triangulated surfaces without boundaries. These kinds of surfaces are called BB 2-manifolds for short. For an oriented BB 2-manifold $\mathcal{M}$ of genus g, it is well known that its fundamental group can be generated by a generating set of $2g$ loops. We denote by M_1 the 1-skeleton of $\mathcal{M}$, i.e., M_1 is composed of the vertices and edges of $\mathcal{M}$. For a subset S of M_1, we denote by $\mathcal{M} \setminus S$ the cut version of $\mathcal{M}$ along S, i.e., $\mathcal{M} \setminus S$ is closed and obtained by cutting $\mathcal{M}$ along S. $\mathcal{M} \setminus S$ is called a polygonal schema, corresponding to S, of $\mathcal{M}$ if $\mathcal{M} \setminus S$ is homeomorphic to a closed disk (see Fig. 3(a)). S is called a cut graph of $\mathcal{M}$ if $\mathcal{M} \setminus S$ is a polygonal schema. A polygonal schema $\mathcal{M} \setminus S$ is said to be canonical if S is composed of $2g$ cycles with properties such that the $2g$ cycles have a common vertex and there is no other common point between any two cycles of S. As defined in [1], S is called a system of loops if $\mathcal{M} \setminus S$ is a canonical polygonal schema.

Polygonal schemas play a very important role in some problems of $\mathcal{M}$, such as surface parameterization and texture mapping, which require information about the underlying topological structure in addition to the geometry. In some cases, we wish to simplify the surface topology in order to facilitate the use of algorithms, since these algorithms can be performed only if the surface is a topological disk. In the texture mapping problem, we wish to find a continuous and invertible mapping from the texture, usually a two-dimensional rectangular image, to the surface, although a global method is possible [2]. Unfortunately, if the surface is not a topological disk, no such map exists. In such cases, one needs to cut the surface so that

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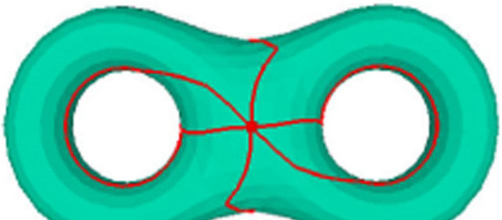
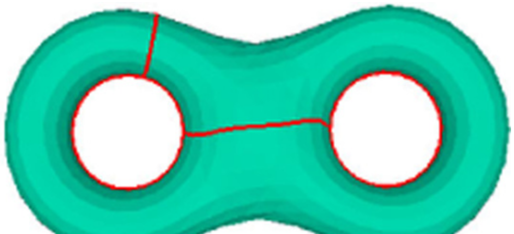


(a) The shortest homology class.



(b) The shortest cycles.

Fig. 1.

it becomes a topological disk. Of course, when cutting the surface, one would like to find the best possible cutting under various conditions. For example, one might cut the surface by the homology basis cycles so that the resulting surface can be mapped with minimum distortion. Erickson/Har-Peled [1] invested the problem of how to find the best polygonal schema $\mathcal{M} \setminus S$, i.e., the length of S is the smallest possible if the edges of M_1 are weighted (Correspondingly, the number of edges of S is the smallest possible if the edges of M_1 are unweighted).

The following problem have been studied by several authors during the last decade [3–6].

Shortest cycles for a canonical polygonal schema: Given a g-genus surface mesh $\mathcal{M}$ and a set S of $2g$ cycles such that $\mathcal{M} \setminus S$ is a canonical polygonal schema, the problem of shortest cycles for a canonical polygonal schema is to find one of the shortest cycles in the homotopy class of each cycle of S (Fig. 1(a)).

In some applications, the request that all cycles of S have a same common vertex may result in some unexpected results.

For example, NURBS (Non-Uniform Rational B-Splines) is not only a de facto standard throughout the CAD/CAM/CAE, but it is incorporated into several international and American national standards such as IGES, STEP and PHIGS [7]. The quality of a NURBS surface is heavily dependent on the degrees of vertices of the quadrangulation, where the degree of a vertex is the number of quads (of the quadrangulation) of sharing this vertex. The high degree vertices may result in unexpected geometric properties for NURBS [8]. But, for the quadrangulations based on canonical polygonal schemas, it is inevitable to produce high degree vertices.

Other disadvantages for a canonical polygonal schema, as mentioned in [4,1], include the following: the shortest loop homotopic to a simple loop may itself not be simple and the size of S can be almost as the same size as M_1. In addition, it is worth pointing out that there exists no canonical polygonal schema if there is no vertex of the triangulated surface such that the degree of this vertex is not less than $2g$.

Therefore, in this paper, we study the following fundamental problem:

Computing a tight orthogonal homotopic basis of $\mathcal{M}$: Given a g-genus surface mesh $\mathcal{M}$, its homotopic basis $\{\alpha_1, \alpha_2, \ldots, \alpha_{2g}\}$ is said to be orthogonal if $\alpha_i \bigcap \alpha_j = \emptyset$ when $|i - j| > 1$ and any two adjacent cycles have exactly one common point, i.e., there exists a point $p_i \in \mathcal{M}$ such that $\alpha_i \bigcap \alpha_{i+1} = p_i$, $1 \le i \le 2g - 1$ (see Fig. 8(b)). An orthogonal homotopic basis S of $\mathcal{M}$ is said to be tight if any cycle of S is one of the shortest cycles in the homotopic class of this cycle. Sure, $\mathcal{M} \setminus S$ is a polygonal schema if S is a tight orthogonal homotopic basis of $\mathcal{M}$.

The solving of this problem can benefit many important applications in graphics. Examples include computing homotopy groups, parameterization, shape retrieval, texture mapping, NURBS surface construction and so on.

2. Previous and related results

Dey and Schipper [9] described a linear-time algorithm that first constructs an arbitrary cut graph G by depth-first search, then shrinks a spanning tree of G to a single point, and finally constructs a canonical polygonal schema for any triangulated orientable manifold without boundary. Vegter and Yap [10] developed an algorithm to construct a canonical schema in optimal $\mathcal{O}(gn)$ time (g is the genus of the surface). Two simpler algorithms with the same running time were later developed by Lazarus et al. [3]. Given a triangulated manifold and a system of loops as the input, Colin de Verdiére and Lazarus [4] developed an algorithm for computing the shortest system of loops in the same homotopic class, in polynomial time under some mild assumptions about input geometry. As a byproduct, they also obtain a polynomial-time algorithm to construct the minimum-length simple loop homotopic to a given path.

Erickson and Har-Peled [1] considered the problem of optimally cutting a surface into a single topological disk along a cut graph of minimum total length. They showed that computing minimum-length cut graph is NP-hard, by a reduction from the classical Steiner tree problem. They described an algorithm with running time $n^{\mathcal{O}(g+k)}$ to find an approximate minimum-length cut graph and also developed a greedy algorithm that computes a $\mathcal{O}(\log^2 g)$-approximation of the shortest cut graph in $\mathcal{O}(g^2 n \log n)$ running time, where n is the combinatorial complexity, g is the genus, and k is the number of boundary components of the input surface.

Erickson and Whittlesey [5] described a simple greedy algorithm to construct the shortest set of loops that generates either the fundamental group (with a given basepoint) or the first homology group (over any fixed coefficient field) of

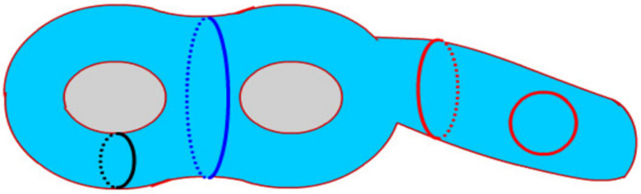


Fig. 2. From left to right: non-separating (black), non-contractible but separating (blue), and trivial cycles (red). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

any oriented 2-manifold. In particular, they showed that the shortest set of loops that generate the fundamental group of any orientable combinatorial 2-manifold, with a given basepoint, can be constructed in $\mathcal{O}(n \log n)$ using a straightforward application of Dijkstra's shortest path algorithm. Based on the universal covering space (UCS) in algebraic topology, Gu et al. [6] consider the same problem by utilizing a different generalization of the Dijkstra's shortest path algorithm.

Colin de Verdière and Erickson [11] obtained an algorithm to compute the shortest path of being homotopic to a given path or the shortest cycle of being homotopic to a given cycle on an orientable triangulated surface. Firstly, they construct a tight octagonal decomposition, i.e., a set of simple cycles, each as short as possible in its homotopic class, that decomposes the surface into a complex of octagons in $\mathcal{O}(n^2 \log n)$ time. After the surface is preprocessed, they compute the shortest path homotopic to a given path of complexity k in $\mathcal{O}(gnk)$ time, or the shortest cycle homotopic to a given cycle of complexity k in $\mathcal{O}(gnk \log(nk))$ time. Cabello et al. [12] described an algorithm to compute a single tight, non-contractible, simple cycle on a given orientable triangulated surface in $\mathcal{O}(n \log n)$ time, and as a consequence they can compute the shortest cycle homotopic to a chosen boundary cycle in $\mathcal{O}(n \log n)$ time and a tight octagonal decomposition in $\mathcal{O}(gn \log n)$ time.

3. Topological background

Here, we introduce some topological terminologies. For a more thorough introduction, refer to Hatcher [13] or Stillwell [14].

A 2-manifold $\mathcal{M}$ is defined to be a topological space in which every point has a neighborhood homeomorphic to $\Re^2$ or to a close halfplane. A boundary point in $\mathcal{M}$ is a point with the property that no neighborhood is homeomorphic to $\Re^2$. The boundary of $\mathcal{M}$ is the union of all boundary points, and it is known to consist of a finite number (possibly 0) of connected components, and each component is homeomorphic to a circle. A manifold is non-orientable if it contains a subset homeomorphic to the Mőbius band, and orientable otherwise. A manifold where every infinite sequence of points has a convergent subsequent is called a compact manifold.

Curve: As usual, a curve or path in $\mathcal{M}$ is a continuous mapping $p : [0, 1] \to \mathcal{M}$, an arc is a path whose endpoints are on the boundary of $\mathcal{M}$, a loop is a closed path, and a cycle is a non-self intersected loop.

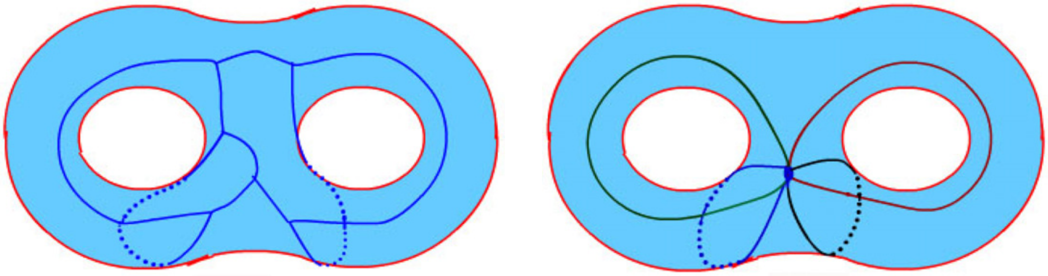
Two curves p, q are said to be homotopic if there is a continuous function $h : [0, 1] \times \mathcal{M} \to \mathcal{N}$ such that $h(0, \cdot) = p(\cdot), h(1, \cdot) = q(\cdot)$, where both $\mathcal{M}$ and $\mathcal{N}$ are 2-manifolds. Such function h is called a homotopy from p to q and the corresponding equivalence classes are called homotopic classes. A loop is said to be contractible if it is homotopic to a point or a single boundary cycle of $\mathcal{M}$. Cutting along a contractible cycle gives two connected components, and one of them is a topological disk or annulus. An arc is said to be contractible if it is homotopic to a boundary path. An arc or a cycle is said to be separating if $\mathcal{M}$ is separated into disconnected parts by cutting along this curve. Non-separating cycles or arcs are non-contractible, while contractible cycles or arcs are separating.

Homotopy basis: According to the classic results of algebraic topology, the set of homotopically equivalent classes of loops of $\mathcal{M}$ forms a group if $\mathcal{M}$ is compact (for a non-compact 2-manifold, similar conclusions also hold by some compacting process of the manifold, but this is out of the range of this paper). This group is called the fundamental group of $\mathcal{M}$ and is denoted by $\pi_1(\mathcal{M})$. Without loss of generality, we can assume that all the loops have a common point $x \in \mathcal{M}$ and denote $\pi_1(\mathcal{M})$ by $\pi_1(\mathcal{M}, x)$ for convenience. x is called the base point. The identity element of the fundamental group is the homotopic class of contractible loops. Although the fundamental group in general depends on the choice of base point, it turns out that, up to isomorphism, this choice makes no difference if the space $\mathcal{M}$ is connected. Therefore, fundamental groups of the same connected space with different base points are isomorphic.

A homotopic basis is defined to be any set of $2g$ loops whose homotopic classes generate the fundamental group $\pi_1(\mathcal{M}, x)$. Homotopic basis is a generalization of the system of loops (see Fig. 3(b)) studied by Colin de Verdière and Lazarus [4]. Every system of loops is a homotopic basis, but the converse is not true. Homotopic basis can contain (self-)intersection that cannot be removed by homotopy (see Fig. 3(a)).

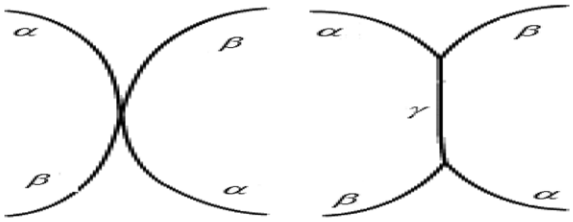
4. Combinatorial surface

Although triangulated surfaces are the main objects of interest here, many conclusions of this paper hold for general combinatorial surfaces. So, we describe our results for combinatorial surfaces. A combinatorial surface is an abstract surface



(a) A homotopic basis but not a system of loops. (b) A system of loops.

Fig. 3. Homotopic bases.



(a) A crossing of a common point. (b) A crossing of a common super path.

Fig. 4. Crossings.

$\mathcal{M}$ together with a graph $G = G(\mathcal{M})$, embedded on $\mathcal{M}$ so that each face of G is a topological disk. Triangulated surfaces are combinatorial surfaces. If G is a graph with V vertices, E edges, and is embedded with F faces, the Euler's formula holds:

$$V - E + F = 2 - 2g - b \tag{1}$$

where g is called the *genus* of $\mathcal{M}$ and b is the number of the boundaries of the surface.

For a path α of G, we define that its multiplicity is the maximum number of times that an edge appears in α. We assume that graph $G(\mathcal{M})$ has edge-weights, which are defined as the Euclidean length of the edges. $|\alpha|$, the length of a path α, is defined as the sum of the weights of the edges in α, counted with multiplicity if they occur on the path more than once. The complexity of a combinatorial surface $\mathcal{M}$ is defined as the sum of the number of vertices, edges and faces of $G(\mathcal{M})$.

A branch point of a cut graph in the combinatorial surface is any vertex with degree greater that 2. A simple path in a cut graph is called a cut path from one branch point or boundary point to another one, with no branch points in its interior. For the combinatorial surface with genus $g > 0$ without boundary, Erickson and Har-Peled [1] proved that any cut graph has at most $4g - 2$ branch points and $6g - 3$ cut paths, with equality if every branch point has exactly degree 3.

5. Finding shortest nontrivial curve

For a combinatorial surface, there are three types of cycles: non-separating, non-contractible but separating, and contractible cycles (see Fig. 2). A contractible cycle is also called a trivial cycle. As can be seen from Fig. 2, it is not possible to determine whether a cycle is non-separating, non-contractible, or trivial, by examining only a local neighborhood.

Making use of Dijkstra's single-source shortest path algorithm in $\mathcal{O}(n \log n)$ time, Erickson and Har-Peled [1] obtained the following results:

Lemma 5.1. *Let $\mathcal{M}$ be a combinatorial surface of genus $g > 0$ with complexity n, possibly with boundary and let u be a vertex in $\mathcal{M}$. The shortest non-separating cycle of $\mathcal{M}$ that contains u can be computed in $\mathcal{O}(n \log n)$ time.*

Lemma 5.2. *Let $\mathcal{M}$ be a combinatorial surface of genus $g > 0$ with complexity n and let α be a boundary cycle of $\mathcal{M}$. Then we can compute the shortest non-separating arc with both endpoints on α in $\mathcal{O}(n \log n)$ time.*

Lemma 5.3. *Let $\mathcal{M}$ be a combinatorial surface of genus $g > 0$ with complexity n and let α and β be two boundary cycles of $\mathcal{M}$. Then we can compute the shortest non-separating arc between α and β with one endpoint on α and the other on β in $\mathcal{O}(n \log n)$ time.*

Consider the k-pairs shortest path problem: Given k-pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of the graph G, compute the shortest path for each pair (s_i, t_i), $i = i, \ldots, k$. Klein [15] and Cabello and Chambers [16] described the problem in planar graph or in a genus $g$graph using the shortest path tree, respectively.

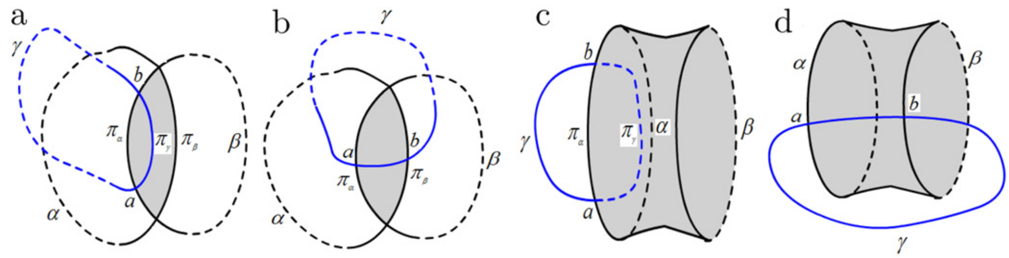


Fig. 5. Pictures used by Lemma 6.2.

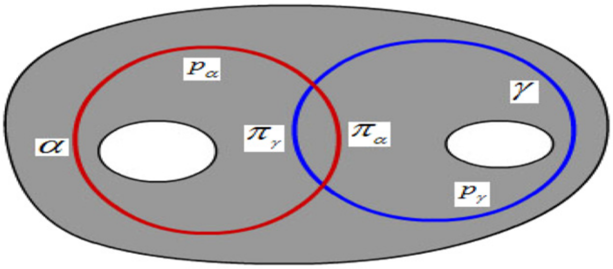


Fig. 6. The bigon is included by α and γ.

Lemma 5.4. 1. *Let G be a planer graph, the k-pairs shortest paths can be solved in* $\mathcal{O}(n\log n)$ *time.*
2. *Let G be a genus g graph, the k-pairs non-separating shortest paths can be solved in* $\mathcal{O}(g^2 n\log n)$ *time.*

6. Tight cycle

A path is tight if it is the shortest possible in its homotopic class. A common path γ of paths α and β is said to be super if none of common paths of α and β contains γ as a proper path. Let γ be a common vertex or a super common path of two cycles α and β. γ is called a crossing (of α and β) if for any neighborhood O of γ, both O_1 and O_2 contain some points of β as interior points, where O_1 and O_2 are the two parts of O separated by α (Fig. 4).

A cycle or arc α is called no crossing by a cycle or arc β, if α keeps being the same cycle or arc in $\mathcal{M}\setminus\beta$. Assuming that the cycles γ,γ' cross each other, we denote by $\mathrm{cr}(\gamma,\gamma')$ the number of all the crossings. Particularly, $\mathrm{cr}(\gamma,\gamma')=0$ if they are no crossing.

Definition 6.1. Let any one of α and β be a cycle or an arc. α and β is said to include a bigon if there are subpaths $p_\alpha\subseteq\alpha$ and $p_\beta\subseteq\beta$ such that p_α and p_β bound a topological disk.

If α and β are homotopic to each other, then α and β bound a cylinder or some disks, so $\mathrm{cr}(\alpha,\beta)$ must be even. Thus,

Lemma 6.1. *Two homotopic cycles α and β in a combinatorial surface $\mathcal{M}$ include a bigon if* $\mathrm{cr}(\alpha,\beta)>0$.

Lemma 6.2. *Let α and β be two homotopic cycles or arcs in a combinatorial surface $\mathcal{M}$ and let γ be a cycle or an arc of $\mathcal{M}$ such that γ is not homotopic to α and β. If γ crosses α and they do not include a bigon, then γ crosses β.*

Proof. Since α and β are homotopic to each other, α and β bound a cylinder if $\mathrm{cr}(\alpha,\beta)=0$ or bound some disks if $\mathrm{cr}(\alpha,\beta)>0$. Assume that γ crosses α to the point a and denote by D the cylinder bounded by α and β if $\mathrm{cr}(\alpha,\beta)=0$ or the disk, bounded by α and β, that takes a as a boundary point (See Fig. 5). Since γ is a cycle or an arc of $\mathcal{M}$ crosses α into D from a, γ has to cross out of D from some boundary point, say b, of D. If $\mathrm{cr}(\alpha,\beta)>0$ and b is on α, then α and γ include a bigon (Fig. 5(a)). This is a contradiction. Therefore, b is on β and γ crosses β to b (Fig. 5(b)). Similarly, if $\mathrm{cr}(\alpha,\beta)=0$ and b is on α, then α and γ include a bigon (Fig. 5(c)). This is also a contradiction. Therefore, b is on β and γ crosses β to b (Fig. 5(d)). □

Lemma 6.3. *Let α be a tight cycle or arc in a combinatorial surface $\mathcal{M}$ and let β be a loop which does not cross α. Then β is tight in $\mathcal{M}\setminus\alpha$ if and only if β is tight in $\mathcal{M}$.*

Proof. If β is tight in $\mathcal{M}$, then it is trivial to prove that β is also tight in $\mathcal{M}\setminus\alpha$. Therefore, we only need to prove that β is tight in $\mathcal{M}$ if β is tight in $\mathcal{M}\setminus\alpha$. If β is not tight in $\mathcal{M}$, then there exists a cycle γ which is tight in $\mathcal{M}$ and homotopic to β such that $|\beta|>|\gamma|$. If γ does not cross α, then γ is also a cycle of $\mathcal{M}\setminus\alpha$. Thus, both β and γ are tight in $\mathcal{M}\setminus\alpha$. Therefore, $|\beta|=|\gamma|$. This is a contradiction. This means that any tight cycle γ of $\mathcal{M}$ crosses α if β is not tight in $\mathcal{M}$. Next, we prove that

γ and α include a bigon. In fact, if γ and α are homotopic to each other in $\mathcal{M}$, according to Lemma 6.1, they include a bigon. If γ and α are not homotopic to each other in $\mathcal{M}$, according to Lemma 6.2, γ and α still include a bigon, since the fact that α does not cross β and the fact that β and γ are homotopic to each other.

Because γ and α include a bigon, there exists subpath $\pi_\gamma \subseteq \gamma$ and $\pi_\alpha \subseteq \alpha$ bound a topological disk, π_γ and π_α are homotopic paths (see Fig. 6). Let $p_\gamma = \gamma \setminus \pi_\gamma$ and $p_\alpha = \alpha \setminus \pi_\alpha$. Let δ_1 be the cycle of p_γ concatenated with π_α. Then δ_1 is homotopic to γ. Since γ is tight, it holds that

$$|p_\gamma| + |\pi_\alpha| = |\delta_1| \geq |\gamma| = |p_\gamma| + |\pi_\gamma| \tag{2}$$

which implies $|\pi_\alpha| \geq |\pi_\gamma|$. Similarly, let δ_2 be the cycle of p_α concatenated with π_γ, then δ_2 is homotopic to α. Since α is tight, it holds that

$$|p_\alpha| + |\pi_\gamma| = |\delta_2| \geq |\alpha| = |p_\alpha| + |\pi_\alpha| \tag{3}$$

which implies $|\pi_\gamma| \geq |\pi_\alpha|$. So we have

$$|\pi_\gamma| = |\pi_\alpha|$$

which implies that

$$|\delta_1| = |\gamma|.$$

If γ and α include other bigons, we can repeat the above processes all the way to obtain a tight cycle δ_1' such that δ_1' is homotopic to β and does not cross α. This is a contradiction. □

Similarly to Lemma 6.3, it holds

Lemma 6.4. *Let β be a loop in a combinatorial surface $\mathcal{M}$ and α be a tight cycle or arc in $\mathcal{M}$. Assume that α and β have at most one common point. Then β is tight in $\mathcal{M} \setminus \alpha$ if and only if β is tight in $\mathcal{M}$.*

Lemma 6.5. *Let α be a shortest non-separating loop or arc based at some point x and let γ be a non-separating arc or cycle that does not cross α. Then γ is tight in $\mathcal{M}$ if and only if it is tight in $\mathcal{M} \setminus \alpha$.*

Generalizing Lemma 6.3, we have the following lemma.

Lemma 6.6. *Let $\alpha_1, \ldots, \alpha_k$ be tight cycles or arcs of a combinatorial surface $\mathcal{M}$ and let β be a cycle or arc that does not cross any one of $\alpha_1, \ldots, \alpha_k$. Then β is tight if and only if β is tight in $\mathcal{M} \setminus (\alpha_1, \ldots, \alpha_k)$.*

Proof. Repeating use Lemma 6.3. □

6.1. Compute tight arcs or cycles

Similarly to the proof of Lemma 6.2, we have the following lemma.

Lemma 6.7. *Let α and β be homotopic cycles or arcs. Then, for any cycle or arc γ, it holds*

$$\mathrm{cr}(\alpha, \gamma) = \mathrm{cr}(\beta, \gamma) \bmod 2.$$

Denote by $\mathrm{Cross}_1(\alpha)$, with respect to $\mathcal{M}$, the set of cycles in $\mathcal{M}$ that cross α exactly once. Then, it holds

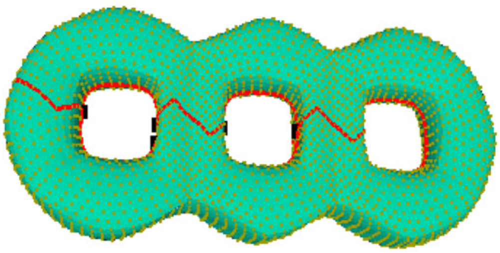
Lemma 6.8. *Let α be a tight cycle or arc, or be a shortest non-separating loop with given base point $x \in \mathcal{M}$. Then, any shortest cycle in $\mathrm{Cross}_1(\alpha)$ is tight.*

Proof. Let β be any shortest cycle in $\mathrm{Cross}_1(\alpha)$. Assume that β is not tight, so there exists a tight cycle γ homotopic to β such that $|\gamma| < |\beta|$. Since $\mathrm{cr}(\alpha, \beta) = 1$, according to Lemma 6.7, $\mathrm{cr}(\alpha, \gamma)$ is odd. Since $|\gamma| < |\beta|$, we have $\gamma \notin \mathrm{Cross}_1(\alpha)$. Therefore $\mathrm{cr}(\alpha, \gamma) \geq 3$. Thus, α, γ include a bigon. As in the later part of the proof of Lemma 6.3, we can construct a cycle $\pi \in \mathrm{Cross}_1(\alpha)$ and $|\pi| < |\beta|$, which is contradictive. This means that β is tight. □

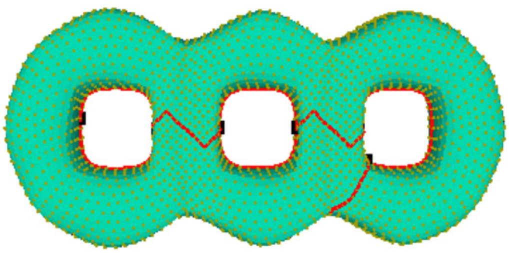
Lemma 6.9. *Let α be a tight cycle or arc in $\mathcal{M}$, or a shortest non-separating loop with given base point $x \in \mathcal{M}$. Then, a shortest cycle in $\mathrm{Cross}_1(\alpha)$ can be obtained in $\mathcal{O}(g^2 n \log n)$ time.*

Proof. In the surface $\mathcal{M} \setminus \alpha$, α produces two copy boundary cycles α', α'' and each vertex v of α also produces two copy vertices $v' \in \alpha'$ and $v'' \in \alpha''$. A cycle in $\mathcal{M}$ that crosses α exactly once to a point $v \in \alpha$ becomes a path in $\mathcal{M} \setminus \alpha$ connecting v' to v'', and vice versa. So, according to Lemma 5.4, we can get the shortest paths of all the pairs $(v', v''),\ v \in \alpha$ in $\mathcal{O}(g^2 n \log n)$ time. Comparing all the shortest paths gives the result. □

Note that if α is separating, then $\mathrm{Cross}_1(\alpha) = \emptyset$ because any cycle crosses α an even number of times. So if α is non-separating, any shortest cycle of $\mathrm{Cross}_1(\alpha)$ is non-separating.



(a) Approximate minimum cut graph.



(b) The shortest homology basis.

Fig. 7. Pseudo-tight and tight homology bases.

7. Construction of a tight orthogonal homotopic basis

For a closed combinatorial surface $\mathcal{M}$ of genus g with graph $G = G(\mathcal{M})$, we now describe a greedy algorithm to construct a tight orthogonal homotopic basis HomoC $= \{\alpha_1, \dots, \alpha_{2g}\}$. For a subgraph K of G, a path of G is said to be pseudo-tight if it is tight in $\mathcal{M} \setminus K$. To obtain a tight orthogonal homotopic basis, we first produce a so-called pseudo-tight cut graph S of $\mathcal{M}$, i.e., S is a cut graph that is composed of $2g$ pseudo-tight arcs and/or cycles (see Fig. 7(a)). Second, we construct a tight homotopic basis based on this tight cut graph (see Fig. 7(b)).

The constructing of a pseudo-tight cut graph. We construct a pseudo-tight cut graph of the combinatorial surface $\mathcal{M}$ of genus $g > 0$ by the following steps:

Step 1: First, according to Lemma 5.1, we select a base vertex and compute a non-separating shortest cycle β_1 in $\mathcal{M}$ with $\mathcal{O}(n \log n)$ running time. Then, β_1 is tight and produces two boundary cycles β_1' and β_1'' in $\mathcal{M} \setminus \beta_1$. Using Lemma 5.3, we can compute a tight non-separating arc β_2 with one endpoint on β_1' and the other on β_1'' in $\mathcal{M} \setminus \beta_1$ with $\mathcal{O}(n \log n)$ running time. Thus, β_2 is a pseudo-tight arc with both endpoints on β_1 in $\mathcal{M}$. If $g = 1$, then we have already obtained a pseudo-tight cut graph $\{\beta_1, \beta_2\}$. If $g > 1$, we continue with step 2.

Step 2: Let $\mathcal{M}_1 = \mathcal{M} \setminus (\beta_1, \beta_2)$. Then $\tau = \beta_1 \cup \beta_2 \cup \overline{\beta_1} \cup \overline{\beta_2}$ is the boundary of $\mathcal{M}_1$, where $\overline{\beta_1}$ and $\overline{\beta_2}$ are the paths of the opposite directions of β_1 and β_2, respectively. Using Lemma 5.3, we can compute a tight non-separating arc β_3 with one endpoint on β_1 and the other on $\overline{\beta_1}$ or with one endpoint on β_2 and the other on $\overline{\beta_2}$ in $\mathcal{M}_1$ with $\mathcal{O}(n \log n)$ running time. β_3 is also a pseudo-tight arc in $\mathcal{M}$ with both endpoints on β_1 or both endpoints on β_2. According to Lemma 6.4, β_3 is tight in $\mathcal{M}$ if it is a cycle in $\mathcal{M}$. Likely, β_3 produces two boundary cycles β_3' and β_3'' in $\mathcal{M}_1 \setminus \beta_3$ and we can compute a tight non-separating arc β_4 with one endpoint on β_3' and the other on β_3'' in $\mathcal{M}_1 \setminus \beta_3$ with $\mathcal{O}(n \log n)$ running time. β_4 is a pseudo-tight arc in $\mathcal{M}$. According to Lemma 6.4, β_4 is tight in $\mathcal{M}$ if it is a cycle in $\mathcal{M}$. If $g = 2$, then we have get a pseudo-tight cut graph $\{\beta_1, \beta_2, \beta_3, \beta_4\}$ and stop.

Step 3: Repeat step 2 $g - 2$ times and denote by $\mathcal{M}_i = \mathcal{M}_{i-1} \setminus (\beta_{2i-1}, \beta_{2i}),\ i = 2, \dots, g-1$. Then, $\mathcal{M}_i$ has genus $g - i$ and the boundary $\beta_1 \cup \beta_2 \cup \overline{\beta_1} \cup \overline{\beta_2} \cup, \dots, \cup \beta_{2i-1} \cup \beta_{2i} \cup \overline{\beta_{2i-1}} \cup \overline{\beta_{2i}}$. Taking each cycle β_k or $\overline{\beta_k}$ as a point, say t_k or $\overline{t_k},\ k = 1, \dots, 2i$, respectively, we can compute $2i$ shortest non-separating arcs for pairs $(t_k, \overline{t_k}),\ k = 1, \dots, 2i$, according to Lemma 5.4, with at most $\mathcal{O}(g^2 n \log n)$ time. Comparing these $2i$ shortest arcs, we select the shortest one (or one of the shortest ones) to produce a new tight non-separating arc, say β_{2i+1}, of $\mathcal{M}_i$ with one endpoint on β_k and the other one on $\overline{\beta_k}$ for some k. β_{2i+1} is pseudo-tight in $\mathcal{M}$ with both endpoints on some β_k. According to Lemma 6.4, β_{2i+1} is tight in $\mathcal{M}$ if it is a cycle in $\mathcal{M}$. Similarly, β_{2i+1} produces two boundary cycles β_{2i+1}' and β_{2i+1}'' in $\mathcal{M}_i \setminus \beta_{2i+1}$. We compute a tight non-separating arc β_{2i+2} with one endpoint on β_{2i+1}' and the other on β_{2i+1}'' in $\mathcal{M}_i \setminus \beta_{2i+1}$. β_{2i+2} is pseudo-tight in $\mathcal{M}$ with both endpoints on β_{2i+1}. According to Lemma 6.4, β_{2i+2} is tight in $\mathcal{M}$ if it is a cycle in $\mathcal{M}$.

The above shows that we need to compute $2g - 1$ tight arcs or cycles. For each arc or cycle, according to Lemma 5.4, we need at most $\mathcal{O}(g^2 n \log n)$ time. So, the total time is $2g\mathcal{O}(g^2 n \log n)$. Thus, we have the following lemma.

Lemma 7.1. *It takes at most $\mathcal{O}(g^3 n \log n)$ time to construct a pseudo-tight cut graph.*

The constructing of a tight orthogonal homotopic basis. Next, we construct a tight orthogonal homotopic basis based on the pseudo-tight cut graph obtained above.

From the previous subsection, we have obtained a pseudo-tight cut graph $C = \{\beta_i,\ i = 1, \dots, 2g\}$. First, let $\mathrm{Cross}_1(\beta_{2g})$ be the set of cycles that crosses β_{2g} exactly once in $\mathcal{M}_{g-1} \setminus \beta_{2g-1}$. Assume that α_{2g} is one of the shortest ones in $\mathrm{Cross}_1(\beta_{2g})$. According to Lemma 6.8, α_{2g} is tight cycle in $\mathcal{M}$.

Next, for each i from $2g$ to 2, let $\alpha_{i-1} = \beta_i$ if β_i is a cycle. According to Lemma 6.4, β_i is tight in $\mathcal{M}$ if it is a cycle. If β_i is not a cycle, let $\gamma_1, \dots, \gamma_k$ be arcs or cycles of $\{\beta_1, \dots \beta_{i-1}\}$ that do not cross with α_i. Let α_{i-1} be one of the shortest cycles in $\mathrm{Cross}_1(\alpha_i)$, where $\mathrm{Cross}_1(\alpha_i)$ is the set of cycles, in $\mathcal{M} \setminus (\gamma_1, \dots, \gamma_k, \alpha_{i+1}, \dots, \alpha_{2g})$, that cross α_i exactly once. Again, according to Lemmas 6.3 and 6.8, α_{i-1} is tight in $\mathcal{M}$. Therefore we can construct a family of $2g$ tight cycles $\{\alpha_1, \dots, \alpha_{2g}\}$. Since $\alpha_{i-1} \in \mathrm{Cross}_1(\alpha_i)$, α_{i-1} and α_i has only one common point, $2 \le i \le 2g$. Note that $\mathrm{Cross}_1(\alpha_i)$ is a set of cycles of $\mathcal{M} \setminus (\gamma_1, \dots, \gamma_k, \alpha_{i+1}, \dots, \alpha_{2g})$, α_j and α_i has no any common point if $|i - j| > 1$.

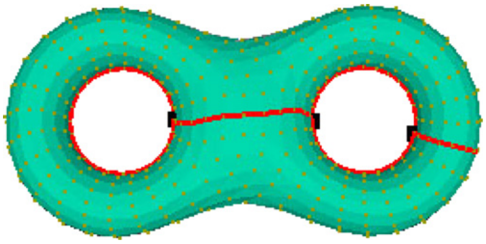


Fig. 8. Cut graph of all the cycles.

In particular, if all $\beta_i,\ i = 1\ldots, 2g$ are cycles (Fig. 8), we only need to compute $\mathrm{Cross}_1(\cdot)$. For this best possible case, according to Lemma 5.4, we need $\mathcal{O}(g^2 n \log n)$ time to construct a tight orthogonal homotopic basis based on a pseudo-tight cut graph. In the worst case, i.e., all $\beta_i,\ i = 2, \ldots, 2g$ are arcs, we need additional $(2g-1) * \mathcal{O}(g^2 n \log n) = \mathcal{O}(g^3 n \log n)$ running time to construct a tight orthogonal homotopic basis from a tight cut graph. Therefore we obtain:

Theorem 7.1. *Let M be a combinatorial surface with complexity n and genus g. Then we can compute a tight orthogonal homotopic basis* $\{\alpha_i,\ i = 1, \ldots, 2g\}$ *in* $\mathcal{O}(g^3 n \log n)$ *time.*

8. Conclusion and future work

In this paper, a method with $\mathcal{O}(g^3 n \log n)$ computing time of constructing tight orthogonal homotopic bases is presented. A tight orthogonal homotopic basis is a cut graph with the following properties.

1. The elements of this basis are cycles.
2. Any two adjacent cycles of this basis have exactly one common point.
3. Any two nonadjacent cycles of this basis have no common point.
4. Any cycle of this basis is one of the shortest cycles of its homotopic group.

In addition, our algorithm can be simplified based on some hierarchical methods. That is, we first construct a nest, from coarser to finer, of combinatorial surfaces that approximate the original combinatorial surface. Then, we start to compute a tight orthogonal homotopic basis for the coarsest combinatorial surface and followed by, based on this tight orthogonal homotopic basis, constructing a tight orthogonal homotopic basis for the nearest finer combinatorial surface, step by step, all the way to construct a tight orthogonal homotopic basis for the original combinatorial surface. We will explore hierarchical methods in further work. In addition, how to find a tight orthogonal homotopic basis with shortest boundary is certainly an interesting and important problem.

Finally, we want to mention that tight orthogonal homotopic bases play a very important role in constructing NURBS surfaces. This will also be one of our future studies.

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